# The Human Group Optimization (HGO) algorithm: Exploiting the collective intelligence of human groups as an optimization tool for complex landscapes.

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**Abstract.** We propose a novel optimization algorithm particularly suited for solving combinatorial non-polynomial problems. The algorithm is inspired by human groups and imitates the way they collectively solve complex problems driven by self-interest and consensus seeking. The HGO describes the statistical evolution of the individual choices within the framework of continuous-time Markov processes. The dynamics of such a system is characterized by a phase transition, from low to high values of the consensus. We recognize this phase transition as being associated with the emergence of a collective superior intelligence of the group. We compare the proposed algorithm with other optimization methods, and, in particular, with the simulated annealing (SA), the multi-agent simulated annealing (MASA) and the genetic algorithm (GA). The efficacy of these methods in solving combinatorial problem defined by *NK* Kauffmann objective function is studied. We show that the HGO method very significantly outperforms the other methods notably in presence of limited knowledge of the agents.

**Keywords:** optimization algorithm; swarm intelligence; artificial Intelligence, decision-making, social interactions, complexity, Markov chains

## 1 Introduction

Solving operational optimization problems often requires the use of heuristic optimization techniques, such as genetic algorithms, simulating annealing, tabu search, to name a few. Recently, a particular class of optimization algorithms, inspired by the complex behavior of natural systems, has become very popular (Conradt and Roper 2003, Brambilla et al. 2013). These algorithms belong to the class of swarm intelligence methods and exploit the potential of the collective decision making (Cheng et al. 2015) in solving complex problems. Animals in groups, e.g. flocks of birds, ant colonies, and schools of fish, exhibit collective intelligence when performing different tasks as, for example, which direction to travel in, foraging, and defense from predators (Conradt and Roper

2003, Couzin et al. 2005). Artificial systems such as groups of robots behaving in a self-organized manner show superior performance in solving their tasks, when they adopt algorithms inspired by the animal behaviors in groups (Krieger et al. 2000, Rubenstein et al. 2014, Werfel et al. 2014). Human groups, such as organizational teams, outperform single individuals in a variety of tasks, including problem solving, innovative projects, and production issues (Lee and Lucas 2014, Brummitt et al. 2015, Clément et al. 2013).

The superior ability of groups in solving tasks originates from collective decision-making: agents make choices, pursuing their individual goals by relying on their own level of knowledge and amount of information, and adapting their behavior to the actions of the other agents. Despite single agents may possess limited knowledge, the collective behavior, enabled by the social interactions, leads to the emergence of a superior intelligence of the group (Bonabeau et al. 2000, Vanni, Luković and Grigolini 2011, Easley and Kleinberg 2010).

Moving from a recent model of human decision-making, recently presented by Carbone and Giannoccaro (2015), we propose a swarm-intelligence-optimization-algorithm, referred to as the Human Group Optimization (HGO) algorithm, to deal with complex combinatorial problems. The algorithm captures the two drivers of the individual behaviors in groups, i.e., the self-interest and the consensus seeking. In making decisions each individual attempts to increase the perceived fitness, which is an estimation of the real fitness value based on individual's knowledge, (Turalska and West 2014, Conradt and Roper 2003) and strengthen the agreement with the individuals (DiMaggio and Powell 1983), whom he/she interacts with through social relationships.

A continuous-time Markov chain is proposed to describe the time evolution of the decision-making process. As in more standard optimization techniques (e.g., simulated annealing) the parameters of the model are continuously tuned, during the optimization process, in order to guarantee the convergence towards the global optimum on the fitness landscape. The latter is built within the framework of the NK model (Kauffman and Levin 1987, Kauffman and Weinberger 1989). We define the transition rate of each individual's opinion change as the product of the Ising-Glauber rate (Glauber 1963), which implements the consensus seeking (Ising 1925, Weidlich 1971), and the exponential rate proposed by Weidlich (1991), which models the self-directed behavior of the individual. We compare our result with those of other optimization techniques in terms of efficacy in solving the NP complete problem defined on the NK Kaufmann landscape, for $K > 2$.

The paper is organized as follows. First, we present the HGO optimization algorithm. We then illustrate the simulation analysis and compare the performance of the algorithm proposed with that of the SA, MASA, and GA. We end with some discussions and conclusions.

## 2 The optimization algorithm mimicking the decision making process of human groups

We consider a human group made of *M* socially interacting members, which has to solve a complex combinatorial problem that consists in identifying the set of decisions (choice configuration) with the highest fitness. The fitness function constructed by employing the *NK* model (Kauffman and Levin 1987, Kauffman and Weinberger 1989). A *N*-dimensional vector space of decisions is considered, where each choice configuration is represented by a vector **d** = $\left(d_1, d_2, ..., d_N\right)$. Each decision is a binary variable taking only two values +1 or –1, i.e. $d_i = \pm 1$ $i = 1, 2, ..., N$. The total number of decision vectors is $2^N$. Each vector **d** is associated with a certain fitness value $V(\mathbf{d})$ computed as the weighted sum of *N* stochastic contributions $0 \le W_j\left(d_j, d_1^j, d_2^j, ..., d_K^j\right) \le 1$ depending on the value of the decision $d_j$ itself and the values of other *K* decisions $d_i^j$ $i = 1, 2, ..., K$. The fitness function of the group is

$$V(\mathbf{d}) = \frac{1}{N} \sum_{j=1}^{N} W_j\left(d_j, d_1^j, d_2^j, ..., d_K^j\right) \qquad (1)$$

The integer index $K = 0, 1, 2, ..., N-1$ represents the number of interacting decision variables, and tunes the complexity of the problem. The complexity of the problem increases with *K*. Note that, for $K \ge 2$, in computational complexity theory, finding the optimum of the fitness function $V(\mathbf{d})$ is classified as a *NP*-complete decision problem (Weinberger 1996). This makes this approach particularly suited in our case.

We model the level of knowledge of the *k*-th member of the group ( $k = 1, 2, ... M$ ) by defining the competence matrix **D**, whose elements $D_{kj}$ take the value $D_{kj} = 1$ if the member *k* knows that the decision $j$ contributes to the total fitness *V*, otherwise $D_{kj} = 0$. Based on the level of knowledge each member *k* computes his/her own perceived fitness (self-interest) as follows:

$$V_k(\mathbf{d}) = \frac{\sum_{j=1}^{N} D_{kj} W_j\left(d_j, d_1^j, d_2^j, ..., d_K^j\right)}{\sum_{j=1}^{N} D_{kj}} \qquad (2)$$

Note that if $\sum_{j=1}^{N} D_{kj} = 0$ the perceived fitness is set to zero. Each member of the group makes his/her choices driven by the self-directed behavior, which pushes him/her to increase the self-interest, and by social interactions, which push the member to seek consensus within the group. When $D_{kj} = 0$ for $j = 1, 2, ... N$, the *k*-th member possesses no knowledge about the fitness function, and his choices are driven only by consensus seeking. Note that the configuration that optimizes the perceived fitness Eq. (2), does not necessarily optimize the group fitness Eq. (1). This makes the mechanism of social interactions, by means of which knowledge is transferred, crucial for achieving high-performing decision-making process.

We build the matrix **D**, by randomly choosing $D_{kj} = 1$ with probability $p \in [0,1]$, and $D_{kj} = 0$ with probability $1-p$. By increasing *p* from 0 to 1 we control the level of knowledge of the members, which affects the ability of the group in maximizing the fitness function Eq. (1). All members of the group make choices on each of the *N* decision variables $d_j$. Therefore, the state of the *k*-th member ( $k = 1,2,.., M$ ) is identified by the *N*-dimensional vector $\boldsymbol{\sigma}_k = \left(\sigma_k^1, \sigma_k^2, ...\sigma_k^N\right)$, where $\sigma_k^j = \pm 1$ is a binary variable representing the opinion of the *k*-th member on the *j*-th decision. For any given decision variable $d_j$, the individuals *k* and *h* agree if $\sigma_k^j = \sigma_h^j$ otherwise they disagree. Within the framework of Ising's approach (Ising 1925), disagreement is characterized by a certain level of conflict $E_{kh}^j$ (energy level) between the two socially interacting members *k* and *h*, i.e. $E_{kh}^j = -J\sigma_k^j \sigma_h^j$, where *J* is the strength of the social interaction. Therefore, the total level of conflict on the entire set of decisions is given by:

$$E = -\sum_j \sum_{(k,h)} J\sigma_k^j \sigma_h^j \qquad (3)$$

where the sum on the indexes *k* and *h* is over pairs of adjacent spins (every pair is counted once) and the symbol $(\cdot)$ indicates that *k* and *h* are nearest neighbors.

A multiplex network (De Domenico et al. 2013) with $j$ = 1,2,.., *N* different layers is defined. On each layer, individuals share their opinions on a certain decision variable $d_j$ leading to a certain level of conflict. The graph of social network on the layer $d_j$ is described by the symmetric adjacency matrix $\mathbf{A}^j$ with elements $A_{kh}^j$. The interconnections between different layers represent the interactions among the opinions of the same individual *k* on the decision variables.

In order to model the dynamics of decision-making in terms of a continuous-time Markov process we define the state vector **s** of the entire group **s** = $\left(\sigma_1^1, \sigma_1^2, ...\sigma_1^N, \sigma_2^1, \sigma_2^2, ...\sigma_2^N, ..., \sigma_M^1, \sigma_M^2, ...\sigma_M^N\right)$ of size $n = M \times N$ and the block diagonal adjacency matrix $\mathbf{A} = diag\left(\mathbf{A}^1, \mathbf{A}^2, ..., \mathbf{A}^N\right)$. Now let be $P(\mathbf{s}, t)$ the probability that, at time *t*, the state vector takes the value **s** out of $2^n$ possible states. The time evolution of the probability $P(\mathbf{s}, t)$ obeys the master equation

$$\frac{dP(\mathbf{s},t)}{dt} = -\sum_l w\left(\mathbf{s}_l \to \mathbf{s}'_l\right) P(\mathbf{s}_l, t) + \sum_l w\left(\mathbf{s}'_l \to \mathbf{s}_l\right) P\left(\mathbf{s}'_l, t\right) \qquad (4)$$

with $\mathbf{s}_l = (s_1, ..., s_l, ..., s_n)$, $\mathbf{s}'_l = (s_1, ..., -s_l, ..., s_n)$. The transition rate $w\left(\mathbf{s}_l \to \mathbf{s}'_l\right)$ is the probability per unit time that the opinion $s_l$ flips to $-s_l$ while the others remain temporarily fixed. Recalling that flipping of opinions is governed by social interactions and self-directed behavior, a possible ansatz for the transition rates is

$$w\left(\mathbf{s}_l \to \mathbf{s}'_l\right) = \frac{1}{2}\left[1 - s_l \tanh\left(\frac{\beta J}{M-1}\sum_h A_{lh} s_h\right)\right] \exp\left\{\beta'\left[\Delta V\left(\mathbf{s}'_l, \mathbf{s}_l\right)\right]\right\} \qquad (5)$$

In Eq. (5) $M$ is the number of agents, $J$ is the strength of social interaction. The pay-off function $\Delta V\left(\mathbf{s}'_l, \mathbf{s}_l\right) = \bar{V}\left(\mathbf{s}'_l\right) - \bar{V}\left(\mathbf{s}_l\right)$, where $\bar{V}\left(\mathbf{s}_l\right) = V_k\left(\boldsymbol{\sigma}_k\right)$, is simply the change of the fitness perceived by the agent $k$, when its opinion $s_l = \sigma_k^j$ on the decision $j$ changes from $s_l = \sigma_k^j$ to $s'_l = -\sigma_k^j$. The transition rates $w\left(\mathbf{s}_l \to \mathbf{s}'_l\right)$ have been chosen to be the product of the transition rate of the Ising-Glauber dynamics (Glauber 1963), and the Weidlich exponential rate $\exp\left\{\beta'\left[\Delta V\left(\mathbf{s}'_k, \mathbf{s}_k\right)\right]\right\}$ (Weidlich 1991). The quantity $\beta$ is the inverse of the so-called social temperature and is a measure of the chaotic circumstances which lead to a random opinion change. The term $\beta'$ is related to the degree of uncertainty associated with the information about the perceived fitness (the higher $\beta'$ the less the uncertainty).

To solve the Markov process Eqs. (4) and (5) we employ a simplified version of the exact stochastic simulation algorithm proposed by Gillespie (Gillespie 1976, Gillespie 1977). The algorithm allows generating a statistically correct trajectory of the stochastic process described by Eqs. (4) and (5).

In Carbone and Giannoccaro (2015), it has been shown that the decision making process performs at the best when consensus about the members sets in, i.e. at the critical value of $\beta J$, which makes the system undergo a transition from low to high consensus. This transition was identified as being associated with the emergence of a superior intelligence of the group, i.e. the so-called swarm intelligence.

The proposed optimization algorithm exploits this property of the system, by increasing the parameter $\beta$ during the optimization process, until the threshold value that determines the consensus is reached. Then, to make the group converge to the optimal point on the fitness landscape (i.e., to the configuration with the highest payoff), the parameter $\beta'$ is also incremented during the optimization. This requirement is similar to the continuous decrease of the temperature in SA, and guarantees that the ratio $\beta J / \beta' \to 0$ as during the process. In detail we choose

$$\begin{aligned} &\beta' = \alpha \log(k) \\ &\beta J = \begin{cases} \alpha\, k; & \beta J < (\beta J)_{th} \\ (\beta J)_{th}; & \beta J \geq (\beta J)_{th} \end{cases} \end{aligned} \qquad (6)$$

where $(\beta J)_{th}$ is the threshold value for the collective intelligence to emerge.

## 2.1 Performance measurement

In Carbone and Giannoccaro (2015) two performance measurements of the collective decision making process were considered: 1) the group fitness value and 2) the level of agreement between the members (i.e. the social consensus). To calculate the group fitness value, the vector $\mathbf{d} = (d_1, d_2, ..., d_N)$ needs to be determined. To this end, consider the set of opinions $(\sigma_1^j, \sigma_2^j, ..., \sigma_M^j)$ that the members of the group have about the decision $j$ , at time $t$. The decision $d_j$ is obtained by employing the majority rule

$$d_j = \mathrm{sgn}\left( M^{-1} \sum_k \sigma_k^j \right), \quad j = 1, 2, ..., N \qquad (7)$$

If $M$ is even and in the case of a parity condition, $d_j$ is, instead, uniformly chosen at random between the two possible values $\pm 1$ . The group fitness is, then, calculated as $V[\mathbf{d}(t)]$ and the ensemble average (i.e. the mean over multiple simulation runs) $\langle V(t) \rangle$ is evaluated. The efficacy of the group in optimizing $\langle V(t) \rangle$ is calculated in terms of normalized average fitness $\eta(t) = (\langle V(t) \rangle - V_{\min})/(V_{\max} - V_{\min})$ where $V_{\max} = \max[V(\mathbf{d})]$ and $V_{\min} = \min[V(\mathbf{d})]$.

The consensus of the members on the decision variable $j$ is measured as

$$\chi(t) = \frac{1}{N M^2} \sum_{j=1}^{N} \sum_{kh=1}^{M} \langle \sigma_k^j(t) \sigma_h^j(t) \rangle \qquad (8)$$

Note that $\langle \sigma_k^j(t) \sigma_h^j(t) \rangle = R_{hk}^j(t)$ is the correlation function of the opinions of the members $k$ and $h$ on the same decision variable $j$.

## 3 Simulation and results

In this section we first analyse the performance of the HGO algorithm for the case of a $NK$ landscape with $N = 12$ and $K$ ranging from 5 to 11. A much more complex case is also analyzed with $N = 18$ and $K = 17$ . We also investigate the effect of the size of the group on the performance of the HGO, by making $M$ range from 3 to 15. We assume that the network of social interaction among the $M$ agents is described by a complete graph. Then, we compare HGO with the Simulated Annealing (SA) and with the Multi Agent Simulated Annealing (MASA) for $N = 12 \quad K = 11$ , and $N = 18 \quad K = 17$ . The comparison is show for increasing levels of knowledge $p$ .

Each stochastic process is simulated by generating 50 different realizations, and the ensemble average of the results is calculated. The simulation is stopped at steady-state, i.e., when changes in the time-averages of consensus and pay-off over consecutive time intervals of a given length is sufficiently small.

In Fig. 1 the time-evolution ($i$ is the time iterator) of the HGO performance are reported for $N = 12$ , $K = 5,9,11$ , and different levels of knowledge $p$ ranging from 0.1 to 1. We observe that not depending on the complexity level $K$ and on the level of knowledge $p$, the increase of $\eta(t)$ is always accompanied by simultaneously increase of $\chi(t)$ .

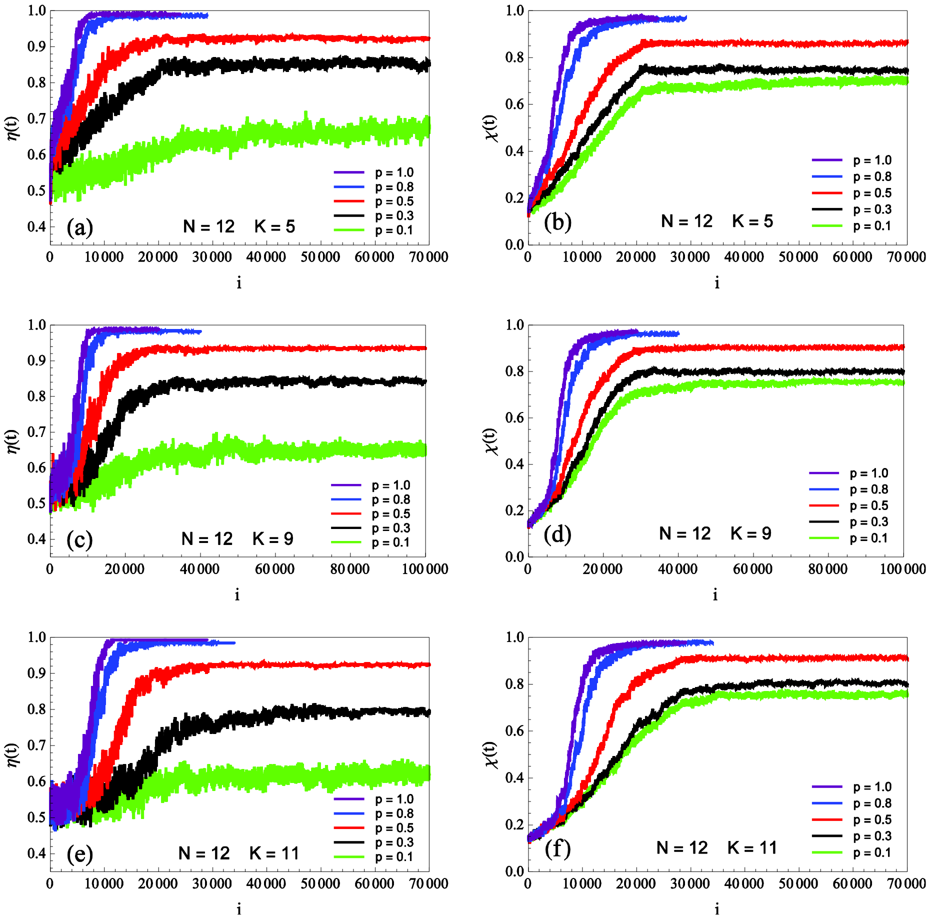


**Fig. 1.** The time-evolution of the normalized average group fitness and consensus for $p$ = 0.1, 0.3, 0.5, 0.8, 1.0, $N$ = 12 and $K$ = 5, 9, 11.

This confirms that the transition from low- to high- level of agreement identifies the emergence of the collective intelligence of the group. Therefore, the occurrence of this transition is a necessary to achieve high performance of the HGO algorithm. Note that the complexity parameter $K$ only marginally affects the performance of the method. The level of knowledge $p$ of the agents, instead, strongly affects the performance of the optimization algorithm, although a high efficacy of the optimization algorithm is already achieved at moderate levels of knowledge, i.e. already for $p$ = 0.5. This can be clearly observed in Fig. 2, where the steady-state values of the efficacy $\eta_\infty$ [Fig. 2(a)], and consensus $\chi_\infty$ [Fig. 2(b)] are plotted as a function of the level of knowledge $p$, for $K = 5, 9, 11$.

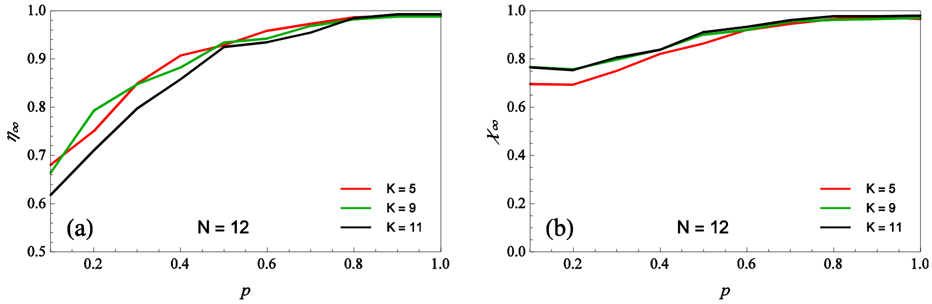


**Fig. 2.** The steady-state efficacy $\eta_\infty$, (a); and degree of consensus $\chi_\infty$, (b); as a function of the knowledge level *p*, for *N* = 12, *M* = 7, and *K* = 5, 9, 11.

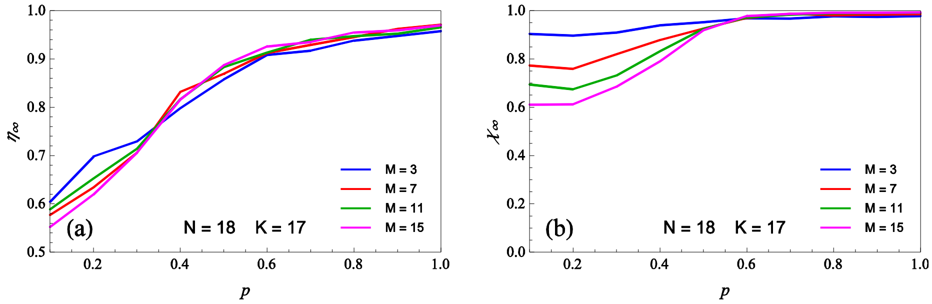


**Fig. 3.** The steady-state efficacy $\eta_\infty$, (a); and degree of consensus $\chi_\infty$, (b); as a function of the knowledge level *p*, for *N* = 18, *K* = 17, and *M* = 3, 7, 11, 15.

Figure 3 shows $\eta_\infty$ and $\chi_\infty$ as a function of *p* for different group sizes $M = 3, 7, 11, 15$, for $N = 18$ and $K = 17$. Note that, for $p > 0.3$, increasing *M* slightly improves the efficacy of the optimization method [see Fig. 3(a)]. In all cases the best results are obtained for $p > 0.5$. Hence, $p = 0.5$ can be identified as a threshold that must be exceeded to guarantee a high degree of consensus [see Fig. 3(b)], and, hence, high fitness values [see Fig. 3(a)].

Fig. 4 compares the HGO with SA and multi-agent simulate annealing (MASA). Results are shown for $N = 12$, $K = 11$ [Fig. 4(a)] and $N = 18$, $K = 17$ [Fig.4(b)], with *p* ranging from 0 to 1. In the case of HGO and MASA, we use $M = 7$.

The HGO algorithm always outperforms the other methods, notably under the condition of limited knowledge of the agents. In this case, the social interaction pushes individuals with no knowledge about a certain decision, to make a good choice as they follow the decisions of those agents, who know the influence of that decision on the fitness value. This type of indirect information sharing is specific of collective intelligence, and makes the entire group perform much better than a group of non-interacting members.

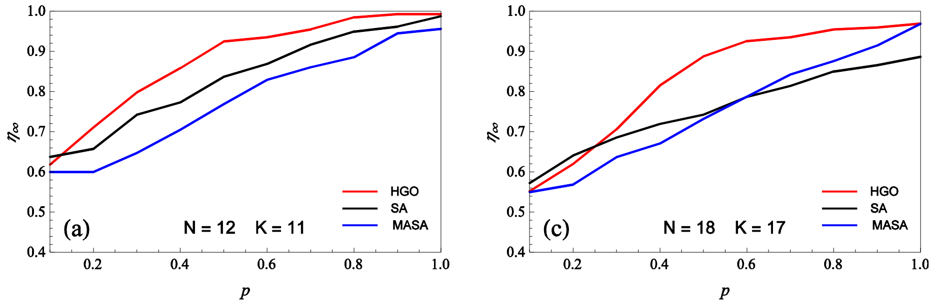


**Fig. 4.** A comparison between the proposed HGO, SA, and MASA, in terms of $\eta_\infty$ as a function of the knowledge level *p*, for *N* = 12, *K* = 11, (a); and *N* = 18, *K* = 17, (b).

| | HGO | SA | MASA | GA |
|---|---|---|---|---|
| N=12, K=1 | 0.999 | 0.856 | 0.996 | 0.888 |
| N=12, K=5 | 0.994 | 0.967 | 0.996 | 0.833 |
| N=12, K=11 | 0.993 | 0.987 | 0.988 | 0.798 |

**Table. 1.** A comparison between the swarm intelligence algorithm (HGO), the simulated annealing (SA) and the genetic algorithm (GA) for, *N* = 12, *K* = 1, 5, 11. The number of agents is *M* = 7 and the level of knowledge is *p* = 1.

Table 1 compares the HGO with SA, MASA and GA. Results are presented for *N*=12 and *K*=1, 5, 11. The number of agents (for the HGO, MASA and GA) is *M* = 7 and the level of knowledge is *p* = 1. We note the in all cases the HGO algorithm always outperforms the other methods and in particular the Genetic Algorithm.

## 4 Conclusions

A swarm intelligence optimization algorithm (HGO) inspired by the collective decision making of human groups is proposed. The algorithm exploits the main drivers that lead to the emergence of a collective intelligence in human groups (i.e., the self-interest of each individual and the search of consensus with the other members of the group) to improve the searching of the optimum on a complex fitness landscape, described by the Kauffman's NK fitness model. Results show that the HGO algorithm outperforms the other methods as the simulated annealing (SA), a multiagent version of the SA and the genetic algorithm. This is even more relevant in presence of limited knowledge of the agents.

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