An Agent-Based Adaptation of Friendship Games: Observations on Network Topologies

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Abstract. A friendship game in game theory is a network game in which a player's immediate neighbors on the network are considered friends. Two friendship-based game models are examined: strategic complements and strategic substitutes. Strategic complements represent decisions for which it is preferable to do what one's friends are doing, such as adopting a common software product. Strategic substitutes represent decisions for which it is preferable to let one friend act alone, such as the private provision of a public good. The game theory models predict the rate of change of preferences and specific equilibrium outcomes over specific time scales for each model. This paper employs an agent-based model (ABM) implementation of friendship games to examine the sensitivity of equilibrium states to network topology. In future work, the ABM model can provide a means to examine the motivations for behaviors of specific individuals in these models beyond closed-form payoff functions.

1 Introduction

Lamberson [7] presents a network-game model of the influence that friends - defined as immediate neighbors on a network - have on individual preferences and the effect this has on long-run equilibrium. Friendship games are applicable to problems for which peer choice is important. Examples include the adoption of standards or common tools, such as word processing software. Additionally, friendship games find use in problems of free-riding, such as the private provision of a public good like a street light or a web server. Lamberson [7] shows that each of these reach one or two distinct equilibrium states. This paper examines how network topology affects those equilibrium states using an agent-based model (ABM) developed in NetLogo [9] for this purpose.[3]

2 Network Games

Galeotti et al. [5] present the theoretical basis for, and some examples of, network games. In these games, the players are distributed on a random network and the payoffs are functions of the expressed preferences of the immediate neighbors on the network. For a model of strategic substitutes, the payoff is such that, if at least one neighbor is paying the cost, none of the other neighbors has an incentive to also pay it. This is a free-rider model, similar to the private provision of a public good. For a model of strategic complements, the payoff is highest for the choice that is supported by a majority of neighbors. This is similar to a network externality, where adopting the most common word processing software, for example, maximizes the ability to share documents with neighbors. Lamberson [7] adopts the term *friend* for these network neighbors, reflecting the fact that adjacent nodes in a social network can be quite distant geographically.

2.1 The Strategic Complements Model

Suppose there are two strategies, x and y. If an agent has k friends, then, at any given instance, there are k_x of them playing strategy x, and k_y of them playing strategy y. For the strategic substitutes models, the payoff for playing strategy x is

$$\pi_x\left(k_x\right) = f\left(k_x\right) - c_x \tag{1}$$

and the payoff for playing strategy y is

$$\pi_y\left(k_x\right) = f\left(k - k_x\right) - c_y \tag{2}$$

where f is a non-decreasing function and c_x and c_y are the costs of playing x and y, respectively.

The adoption of a standard is a strategic complement: an agent chooses what most of its friends choose. A decision to adopt a strategy has a positive affect in that friends tend to take the same choice as the agent.

2.2 The Strategic Substitutes Model

For the strategic substitutes models, the payoff for playing strategy x is

$$\pi_x(k_x) = 1 - c_x \tag{3}$$

where $0 < c_x < 1$ and the payoff for playing strategy y is

$$\pi_y(k_x) = \begin{cases} 1 & k_x \geq 1 \\ 0 & \text{otherwise} \end{cases}. \tag{4}$$

The provision of a public good is a strategic substitute: an agent needn't provide it unless none of its friends do. The decision to adopt a strategy has a negative affect in that friends tend to take the opposite choice of the agent.

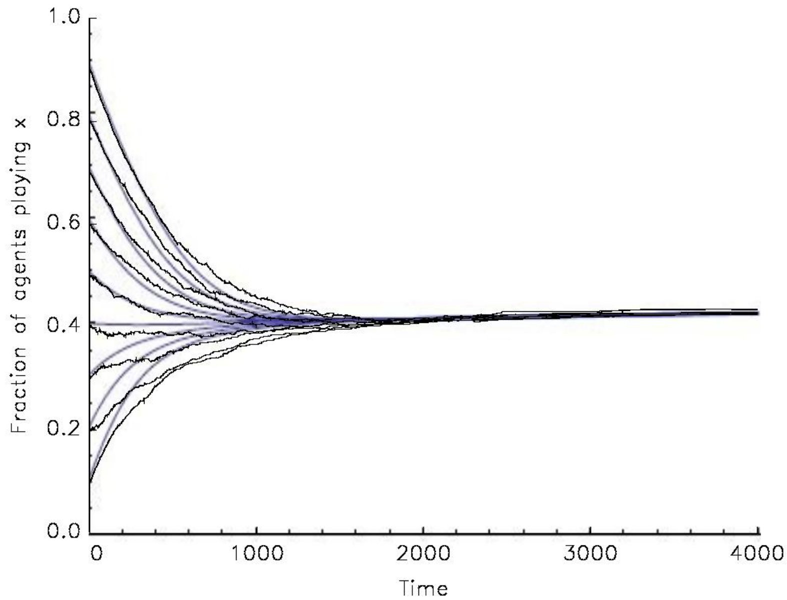
3 Approximating a Random Network

The models in [7] feature 1000 players on a random network. In some cases it is a regular network, in others it is a Bernoulli random network with an edge probability of 0.01. That is, for a network potentially connecting all players to all players, there is a probability of one in one hundred that a given connection will actually be there. The number of other players to which a player is connected is that player's *degree*. The average degree for this random network is approximately 10. That is, players have, on average, ten friends.

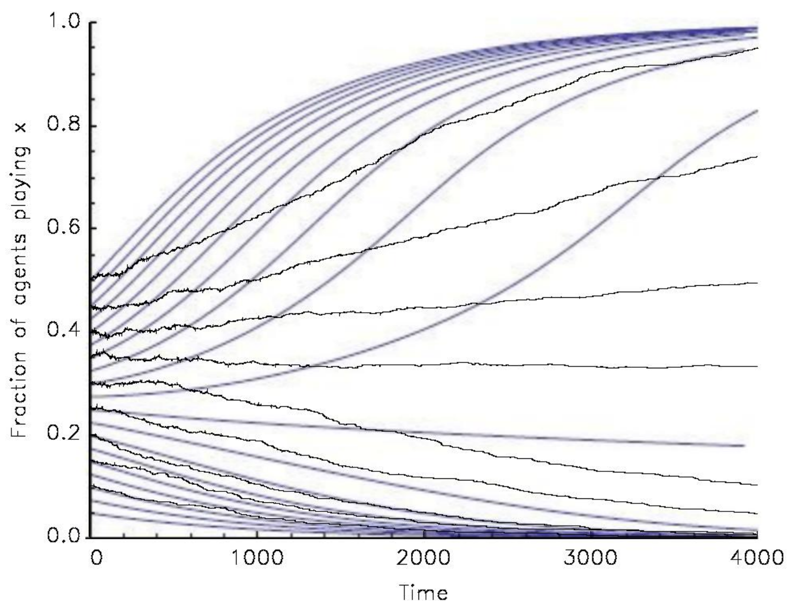
In the ABM developed for this paper, there are fours ways in which a random network can be generated. These are referred to as the *regular*, *Erdős-Rényi*, *Gilbert,* and *preferential attachment* network models. The following are descriptions of these network models.

3.1 The Regular Random Network Model

One way to form a random network is for each agent to make two friends, but only with other agents that don't already have two friends. This results in a regular random network where the nodes



(a) Strategic substitutes, cost $c_x = 0.5$.



(b) Strategic complements, cost ratio $c_x : c_y = 50 : 50$.

Fig. 1: Degree two regular network overlaid on [7] Figs. 2 and 1, respectively.

have a uniform degree of two, ensuring that the average degree is two. It is a high connectivity network: no agents will end up completely disconnected from the network.

For example, a simple form of the strategic substitutes model in Sect. 2.2 is implemented in NetLogo and simulated as outlined in Sect. 3.5. Figure 1a shows the results for a degree 2 regular network of 1000 nodes. This plot overlays the ABM results (black) on an image of the corresponding numerical results in [7] (blue). Three values of the cost shown in Sect. 2.2 are simulated, $c_x = 0.25$, 0.50, and 0.75, all with identical results. The $c_x = 0.50$ results are shown in the figure. As with the numerical result, about 40% of the nodes in the ABM are playing strategy x at equilibrium, and the ABM takes slightly longer to reach equilibrium than the numerical model.

Similarly, a simple form of the strategic complements model in Sect. 2.1 is implemented in NetLogo and simulated as outlined in Sect. 3.5. Figure 1b shows the results for a degree 2 regular network of 1000 nodes. This plot overlays the ABM results (black) on an image of the corresponding numerical results found by [7] (blue). Three ratios of the costs shown in Sect. 2.1 are simulated, $c_x : c_y =$ 25:75, 50:50, and 75:25, all with identical results. The $c_x : c_y =$ 50:50 results are shown in the figure. In the numerical model, an initial distribution of 22.5% or fewer playing x move to an *all-out* (no players playing x) equilibrium. Similarly, initial distributions with 30% or more playing x move to an *all-in* (all players playing x) equilibrium. In the ABM results, an initial distribution of 40% playing x appears to be moving to the all-in equilibrium, an initial distribution of 30% playing x go to the all-out equilibrium, and an initial distribution of 35% playing x is decreasing slowly but monotonically. Here the ABM results differ considerably from the numerical results. The split between the all-in and all-out equilibria is evident but at a higher initial distribution in the ABM, and the time for the ABM to reach equilibrium is much greater. The differences are not yet explained.

3.2 The Erdős-Rényi Random Network Model

A Bernoulli random network with n nodes and probability p that an edge will exist results in nodes with degrees that are binomially distributed about the mean np.[4] There are two common models for this type of network. One is the the *G(n, M)* model, which is called the *Erdős-Rényi random network model* in this paper, and the other is the *G(n, p)* model, which is called the *Gilbert random network model* in this paper (see Sect. 3.3).

In the *G(n, M)* model, a network is chosen at random, with uniform distribution, from the collection of all possible networks with n nodes and M edges. For the models in this paper, M is not known a priori, so a *G(n, M)* model is approximated by adding edges between randomly chosen pairs of nodes until the mean degree reaches the desired value.

3.3 The Gilbert Random Network Model

If all friendship pairs are equally probable with probability p then a Gilbert random network, *G(n , p)* is formed.[6] The mean degree is np, where n is the number of nodes. For the models in this paper, this is created with nested loops: an outer loop over a randomized list of all nodes, and an inner loop over a randomized list of all nodes that haven't already come up in the outer loop. In the inner loop, a connection is formed if a uniform random draw is less than or equal to p.

3.4 The Preferential Attachment Network Model

The preferential attachment model [10] is included in the NetLogo [9] demo library and is based on an approach by Barabási and Albert [2]. This is an approximation of a *scale-free network,* a network with a power-law distribution of node degree. Also called a Pareto distribution, it results in a few nodes having a very large number of connections. Scale-free networks are seen in academic citations [8] and in a variety of Internet linkages.[1] Albert an Barabási [1] found that the probability of a link for a node with degree k is $P(k)\alpha k^{-\gamma}$where γ is between 2 and 3. The NetLogo preferential attachment algorithm yields a γ of approximately 1.4.

3.5 The ABM Simulation

In order to make a direct comparison with the numerical models in [7], the NetLogo models update a single, randomly selected agent at each time step. This random sampling means that, for a network with 1000 nodes, in the first 1000 time steps, some agents may not be updated at all, and others may be updated more than once.

These are the steps in a simulation:

1. Randomly assign agents an initial strategy. Each run involves nine simulations, each with a different initial strategy distribution:
   (a) For strategic substitutes, starting initial distributions of the fraction of agents playing strategy x are 10% through 90% in steps of 10%.
   (b) For strategic complements, starting initial distributions of the fraction of agents playing strategy x are 10% through 50% in steps of 5%.
2. Each time step, a node is selected at random and that node selects a strategy based on the payoffs. This may be the same as the strategy already being played.
3. Each simulation proceeds for 4000 time steps, except as noted.

3.6 The Degree 10 Random Network Models

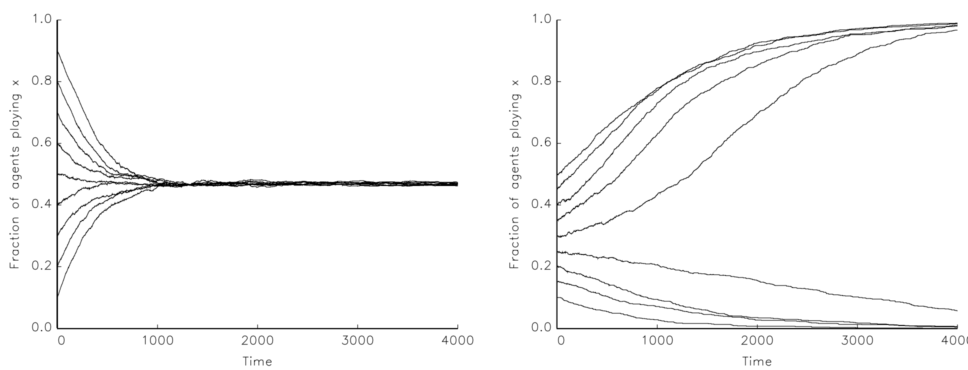
The following models use degree 10 random networks with 1000 nodes and payoff functions suggested in [7]. For a strategic substitute, the payoff for playing x is positive if four or fewer neighbors are playing x, and the payoff is zero otherwise:

$$\pi_x^{substitute} = \begin{cases} 1 & k_x \leq 4 \\ 0 & \text{otherwise} \end{cases}. \tag{5}$$

For strategic complements, the payoff for playing x is positive if four or more neighbors are playing x, and the payoff is zero otherwise:

$$\pi_x^{complement} = \begin{cases} 1 & k_x \geq 4 \\ 0 & \text{otherwise} \end{cases}. \tag{6}$$

Degree 10 Regular Random Network. Plots of the ABM results with a degree 10 regular random network are shown in Fig. 2. The equilibrium for strategic substitutes is between 46.6% and 47.9% playing x. For strategic complements, the all-in or all-out division is between 25% and 30% playing x.

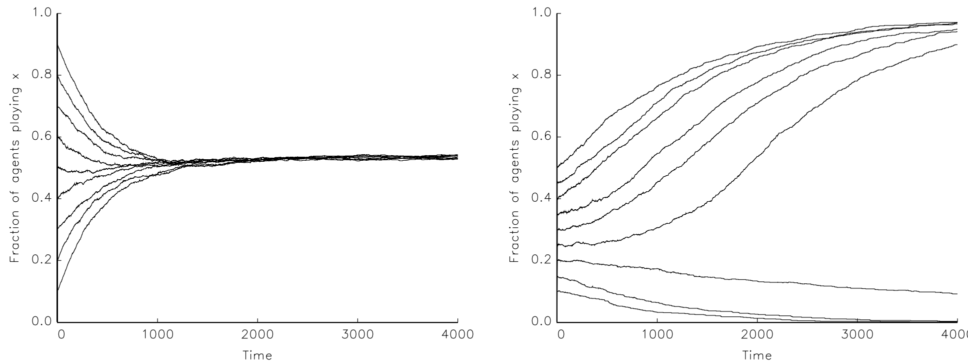


(a) Strategic substitutes. (b) Strategic complements.

Fig. 2: Degree 10 regular random network.

Degree 10 Bernoulli Random Network. Plots of the ABM results with a degree 10 Bernoulli random network are shown in Fig. 3. This is an Erdős-Rényi random network, but the results for a Gilbert random network are effectively identical. The equilibrium for strategic substitutes is between 53.0% and 54.3% playing x. For strategic complements, the all-in or all-out division is between 20% and 25% playing x.

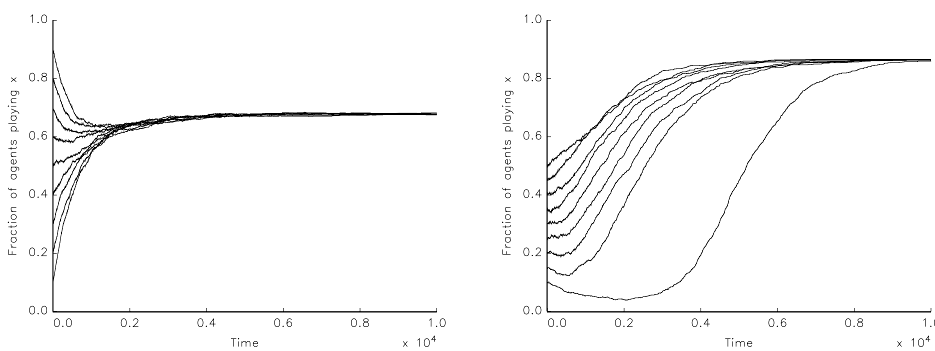
Degree 10 Preferential Attachment Network. Plots of the ABM results with a degree 10 preferential attachment random network are shown in Fig. 4. Note that the horizontal axis goes to 10000 time steps for these plots only. The strategic substitutes curves converge at about 1400 time steps, similar to the other degree 10 random



(a) Strategic substitutes. (b) Strategic complements.

Fig. 3: Degree 10 Bernoulli random network.

networks, but the equilibrium of between 66.8% and 68.4% playing x is not reached until about 6000 time steps. For strategic complements, for 20% and below playing x, the curves are no longer monotonic, starting downward at first, then curving up, ending at the all-in equilibrium.



(a) Strategic substitutes. (b) Strategic complements.

Fig. 4: Degree 10 preferential attachment ABM with $n = 1000$

4 Discussion

The intriguing fact of stable equilibria in friendship games is established numerically in [7]. The correspondence of the ABM to the numerical model is demonstrated in [3]. What is shown in this paper is that the equilibrium values are affected by network topology. For the strategic substitute models, the equilibrium increases with both the mean and the variance in degree, starting at about 40% for a degree 2 regular network, going up to about 68% for a degree 10 power-law network. For the strategic complement models, the division between the all-in and all-out equilibria appears to be decreasing, going from about 25% for a degree 2 regular network to below 10% for a degree 10 power-law network.

Of particular interest with strategic complements and the power-law network is the reversal of the downward trend for the lower curves. This may be an outcome of a high-degree node not being sampled until late in the simulation. The equilibria for the strategic substitute models, for example, show single percentage oscillations late in the simulations, presumably a result of nodes not sampled earlier.

Outcomes peculiar to power-law networks are of interest in the study of Internet phenomena. The recovery of a declining meme and it's ultimate primacy may have implications for Internet hoaxes, viral messages and resurgent Internet memes.

5 Future Work

The payoffs in a friendship game ABM are not constrained to closed-form mathematical functions, and can incorporate adaptive behaviors such as learning and heuristics. These could enable the construction of models of voters, consumers, or decision-makers in which the payoff (or utility or fitness) depends on the preferences of multiple groups of friends over multiple conflicting issues.

In the short-run, further examination of the dependency of equilibria on the distribution of degree is warranted. Also anticipated is an exploration of how the form of a payoff function affects the equilibrium outcome.

6 Conclusion

Network topology has a significant effect on the equilibrium outcome of a friendship game. In particular, for a power-law network, the trend for a declining strategy in a strategic complement friendship game can reverse. This may have implications for the spread of memes on the Internet and elsewhere, for example. Further study of the relationship of network topology to equilibrium outcomes is suggested, as well as expansion of friendship games into adaptive-agent modeling.

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