Measures of Polarization and Diversity – abridged version –

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Abstract. This paper identifies ten distinct ways in which to characterize the concepts of both 'polarization' and 'diversity'. Although these two concept clusters are not typically associated, both are descriptions of the distribution of properties or attitudes (broadly conceived) across a population. In addition to distinguishing and describing the ten senses of polarization and diversity, we offer example formal measures for each one, discuss their relations, and survey the benefits of combined measures. A deeper understanding of the phenomena of polarization and diversity comes from a careful analysis of the distinct senses in which they can occur, and an integration of each distinct sense back into the bigger picture. This formal work can be used to clarify the claims regarding polarization in simulations and empirical data and the match or mis-match between them.[1]

1 Introduction

Neither 'polarization' nor 'diversity' is the single concept it is often taken to be. Occurrences of 'polarization' frequently appear in the sociology and political science literature [1–7], but the specific ways in which polarization is realized are not clearly distinguished. Often entire articles appear on the topic of polarization, but with little attempt to make it clear what precisely is meant by the term. A genuine understanding of the phenomena at issue demands that we do better in making our definitions precise. The same holds for 'diversity', a term broadly applied in a wide variety of senses without a rigorous treatment of what specific property is being described (although, see Page [8]). Here we tease out a variety of distinct concepts commonly bundled with these two terms, and indicate how they can be made precise with example formal measures of each concept.

We recognize that the social phenomena termed 'polarization' and 'diversity' are not often thought of as related. Indeed 'polarization' is generally taken as a negative social measure: the dysfunctional disassociation of major portions of a population, undesirable both in itself and in its effects.'Diversity', on the other hand, has been touted as a positive social measure, argued for both as a social good in itself and in its effects.By focusing on the representation of these phenomena within a distribution of ideas (beliefs, opinions, tastes, attitudes, attributes, properties, etc. all used interchangeably throughout) and on formal

[1] this version has been edited from 20 to 12 pages in order to meet the conference submission length requirement. The full draft is available, appended to this document.

measures of those distributions, we argue that the two concepts are in fact quite intimately mathematically related.

Our first purpose in what follows is to emphasize that both polarization and diversity capture concept clusters rather than single concepts. Without claim to completeness, we distinguish ten distinct senses in which each term can (and has been) used. When appropriate we provide clear formal measures in terms of both 1) distributions of attitudes across an unstructured population and 2) distributions of attitudes across exogenously defined groups of individuals. In the course of that disambiguation we emphasize the relations between diversity and polarization in terms of each measure. In some cases the effect of increase in a measure is the same on diversity and polarization: they increase in parallel. In some cases an increase in a measure has opposite effects on the two. We also briefly consider compound concepts that require combining multiple senses; a consideration that is especially important when addressing a literature that commonly combines or switches the meaning of these terms.

2 Linking Polarization and Diversity

Although polarization and diversity are distinct concepts, and carry different social overtones, they can both be characterized as descriptions of patterns in distributions. Our definitions and example measures below will make clear how the same measure on the same kind of data can be interpreted as polarization, and without any change in the conceptual or mathematical distinctions be interpreted as diversity.

The following guiding principle is used to help clarify the effect that different distributions have on the concept of being polarized.

Definition 1. *General Principle of Polarization: the more effort required to bring people's attitudes into agreement the greater the level of polarization.*

That principle includes a notion of effort, which is purposefully left ambiguous in order to achieve generality. The definitions and measures presented below do not depend on any particular mechanism, nor any gauge of the effort to actually unify attitudes. In other work we identify several mechanisms that do produce polarization in the myriad senses discussed here, but for our purposes here you can imagine any simple mechanism that will help your intuition. Some mechanisms, especially very complicated ones, may reverse the assessment of the level of effort required, but such contrived possibilities are far afield from our purposes in using this principle in ranking different distributions of attitudes as more or less polarized.

Our notion of diversity, however, does not depend on any underlying mechanism or function [8].

Definition 2. *General Principle of Diversity: the more variety in people's attitudes the greater the level of diversity.*

It is also not specifically related to the number of distinct types or categories of objects (such as species, races, religions, political parties, etc.; that is just one sense of diversity). A population is more diverse with respect to a belief, opinion, idea, attitude, etc. the more the distribution of that belief, opinion, idea, attitude, etc. differs within the population.

3 Formal Measures for Polarization and Diversity

The definitions for each of the ten senses below are provided along with example measures. The definitions are conceptual distinctions that stand on their own, while the measures are mathematical means to identify those conceptual distinctions in histograms of belief distributions along a spectrum.[2] Although some of the particular measures provided here will not work in capturing these distinctions in every type of data (e.g categorical or overly coarse-grained data), they should provide clarity on the conceptual distinctions and are applicable to two key forms of data: histograms of attitude distributions from surveys and simulation outputs of models with epistemic agents.

For concreteness, the measures provided here assume the following characteristics of a dataset and representation of the data. There are N agents: $a_1, a_2, \ldots a_N$. Each agent a_i has an attribute value x_i on a range normalized to between 0 and 1; this is the location of that agent's belief, opinion, attitude, etc. along the spectrum. The distribution of all agents' attributes is written X. The set of bins used for discrete characterizations is R. For $r \in R$, the expression $y(x_r)$ is the number of agents occupying the bin containing the value x_r.

3.1 Spread

As already noted, we are using a spectrum of ideas to demonstrate the various senses of polarization and diversity. The simplest concept of polarization and diversity is that of how wide the field of ideas actually represented in the system is. Without taking account of the shape of the distribution of ideas, or even whether there is continuity between the extremes, the further out individuals are the more varied their beliefs and the more difficult is would be to bring them together. Therefore the wider the difference in the most extreme views held, the more diverse and the more polarized the population's ideas are (in this sense).

In a modeled range of beliefs, polarization in the sense of spread could be measured as the belief level of the agent with the highest belief value minus the belief level of the agent with the lowest belief value. This measure is also commonly called the range of the data.

$$\text{spread} = \max_{x_i} X - \min_{x_i} X \tag{1}$$

[2] Specifically, we provide measures that operate on a metric space so that there is a clear ordering from one end to the other and that distance measures are well-defined everywhere.

For multidimensional data, the diameter of the system (the longest pairwise distance among all the points) is the general measure of spread, of which this is a special case.

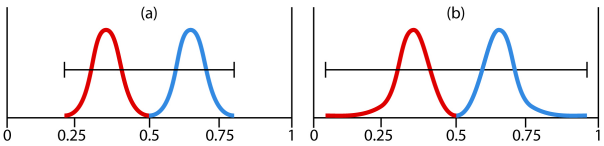

Fig. 1. Belief distribution (b) shows greater polarization and diversity in the sense of spread than does belief distribution (a)

3.2 Dispersion

Another simple, and common, measure of the variation in attributes is statistical dispersion. Unlike spread, which considers only the extremes of the population, dispersion considers the shape of the whole distribution. Dispersion can increase when groups move apart, or when the distribution flattens, or when the agents within the distribution move away from the middle toward opposite ends of the distribution. Just as in spread, greater dispersion implies that it is more difficult to reach a consensus and a great variation in beliefs. Therefore both polarization and diversity increase with increasing dispersion.

There are many measures of statistical dispersion that are appropriate for our demonstration application: mean difference, average absolute deviation, standard deviation, coefficient of variation, and entropy are all candidates. For simplicity we select average absolute deviation from the mean as our example, although we also track the mean difference measure in our application of this measure to data in other work. For a population with N individuals we have

$$\text{dispersion} = \frac{1}{N} \sum_{x_i}^{N} |x_i - \bar{X}| \ , \tag{2}$$

where $\bar{X}$ is the mean value of distribution X.

3.3 Coverage

Polarization and diversity in the sense of coverage captures the level of variation in the values held. One can think of it as the number of distinct attitudes held or the variety of ideas that at least one person in the population has. Although this sense is not sensitive to the shape of the distribution, or even the number of agents who hold each position, it does capture a basic feature of variation. For this sense it does not matter how different the values are, or whether they

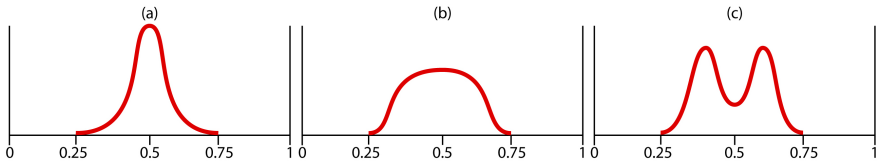

Fig. 2. Distribution (c) shows greater polarization and diversity in the sense of dispersion than does belief distribution (b), which is greater than distribution (a).)

are clumped together or at extremes, it only captures the variety of distinct attributes in the population. It should be clear from that description that diversity increases with increasing coverage. Because more coverage implies more different ideas, and this variety is at odds with converging ideas, polarization is higher as well.

A simple way to measure coverage in a discrete (binned data) spectrum is to simply calculate the proportion of bins that are occupied by at least one agent. Note that this discrete measure does not rely on any distance measure or ordering of the values, and therefore is also appropriate for categorical data.

$$\text{coverage}_d = \frac{1}{R} \sum_{r \in R} \begin{cases} 1 & y(x_r) > 0 \\ 0 & y(x_r) = 0 \end{cases} \tag{3}$$

We also present a continuous measure that does not rely on binning the data. This is done by setting halos of radius r around each agent – for one-dimensional data this makes an agent-centered "bin" of width $2r$. Any portion of the belief space within r of an agent is considered covered, with the rest considered uncovered. Coverage is measured by calculating the proportion of the space covered by those halos.

$$\text{coverage}_c = 2r + \sum_{x_i < x_j}^{X} \begin{cases} x_j - x_i & x_j < x_i + r \\ r + x_j - x_i & x_i + r \leq x_j \leq x_i + 2r \\ 2r & x_j > x_i + 2r \end{cases} \tag{4}$$

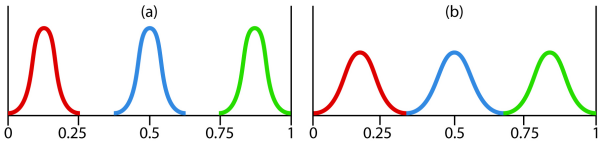

Fig. 3. Distribution (b) is more polarized and more diverse than (a) in the sense of coverage.

3.4 Regionalization

In addition to the area covered by beliefs, patterns in the beliefs *not* held by anybody in the system are also relevant for capturing polarization and diversity. The number of uncovered ideas is the same concept as coverage. However, the number of uncovered intervals (ignoring their size) is a distinct sense of variety, and hence a different way that distributions can be polarized or diverse. This sense captures the idea that if there is no intermediate values between positions, then they are more difficult to bring together, and more distinctive in variety. Thus the more empty regions, the greater the regionalization, and the greater both polarization and diversity are.

In order to measure the regionalization on a discrete spectrum it is sufficient to count the regions of contiguous empty bins (including the initial space and trailing space). One can also define a continuous version of this measure using halos, just as we did for coverage, which better accounts for gaps in dense datasets. For a discrete spectrum with ordered bins R we can calculate the number of gaps with the following formalism:

$$\text{regionalization} = \sum_{r_i < r_j \in R} \begin{cases} 1 & y(r_0) = 0 \\ 1 & y(r_i) > 0 \text{ and } y(r_j) = 0 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

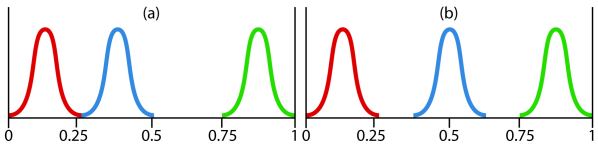

Fig. 4. Distributions with equal coverage and spread, but in which (b) shows a larger number of empty spaces between occupied areas, indicating greater polarization and diversity in the sense of regionalization.

3.5 Community Fracturing

The core of community fracturing is the degree to which the population can be broken into sub-populations. As a sense of polarization the more groups there are, the more difficult it tends to be to get the individuals in those groups to agree, and so the greater polarization is. Diversity also increases with an increasing number of groups because even if each group shares the same belief profile they differ with respect to the group they are in. The number of groups, in and of itself, does not reflect the distribution of beliefs; i.e., holding all descriptions of the size and shape of the distribution constant, more separate groups implies more polarization and more diversity in this sense. The implications this has

on a situation depends on what is meant by "group" in that context. Here we consider three different ways of understanding groups.

The first way is for when there is no information about the social structure underlying a belief distribution. Groups can be identified directly from the histogram as collections of individuals included within the basin of attraction around peaks as in Figure 5 (a) and (b). In this way the groups are identified endogenously by the patterns in belief values. For example, in the attitudes toward abortion in figure 6 we can see a clear group on the far liberal (left) side, and another group in the central position, and*in some years* a third group when there is an upturn on the far conservative (right) side.

The second way to define groups is by exogenous variables. In many cases the data are organized by ethnicity, nationality, sex, or other categories; e.g., opinions on education spending grouped by education attainment level. Such an application can create overlapping sub-populations such as in Figure 5 (c). A third way to identify groups comes from agents on networks, grids, or spatial arrangements. Social, physical, and property relationships can call be the basis for grouping individuals together.

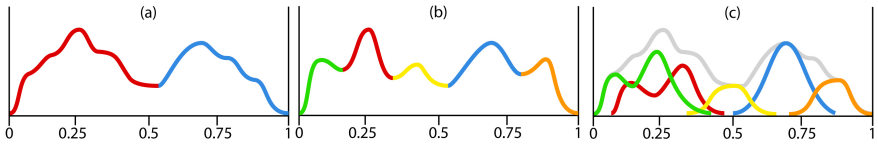

Fig. 5. Polarization and diversity increases from (a) to (b) for endogenously defined groups. The histogram for the entire population may be broken in to varying numbers of sub-populations, as in case (c).

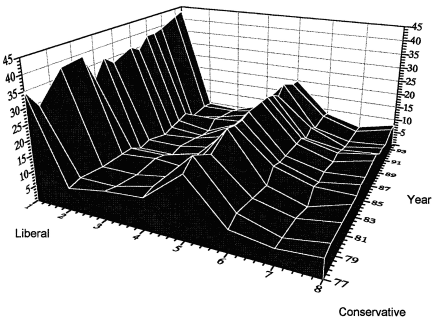

Fig. 6. Attitudes toward abortion, distribution by year, from the full sample General Social Survey 1997-1994 [4].

3.6 Distinctness

After we have identified different belief or property groups whether as peaks along the spectrum or exogenous categories we can ask how different these factions are. What matters for polarization in this sense is how well we can distinguish the two sub-distributions or sub-populations. The more clearly they can be seen as separate, the more polarized the total population is. Greater distinctness, or distinctiveness, also implies greater diversity. On the other end of distinctness, two groups with the same distribution would have minimal distinctness and therefore minimal diversity. By distinctness we expressly mean the degree to which the groups are separate and distinguishable from each other, even regardless of the distance between them.

A simple measure for two groups that is useful for agent-based models of these phenomena is:

$$\text{distinctness}_{\text{exo}} = |A| + |B| - \sum_{r \in R} \begin{cases} |y_A(r) - y_B(r)| & y_A(r),\, y_B(r) > 0 \\ 0 & \text{otherwise} \end{cases} . \quad (6)$$

When there are more than two groups in the population, some aggregation of the pairwise comparisons must be made. For our purposes the mean of all pairwise comparisons is sufficient.

Borders for endogenously identified groups are defined by the local minima in the distribution. The more individuals that exist at this boundary – the ones that are between the two groups – the less distinct the groups are.

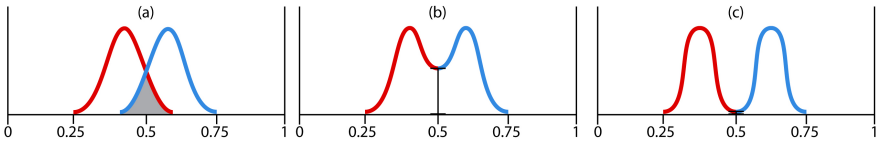

Fig. 7. When measured on epistemic networks, belief Three belief distributions, with $(b) > (a)$ in terms of polarization as distinctness.

3.7 Group Divergence

Group distinctness captures how different groups are with respect to being separate regardless of how far away those groups are in their beliefs. Group divergence captures the reverse: how distant the groups' ideas are without accounting for their shapes. The more the groups diverge the greater the polarization and diversity of the population is because this is another way in which the difference makes convergence more difficult while adding variety to the population's collection of beliefs.

For our current purposes it is sufficient to use the distance between the means of the of the groups (or the area of the convex hull of the means for more than

two groups). This measure applies whether the groups are defined endogenously or exogenously. For a set of G belief sub-distributions (groups) $\{g_1 \dots g_G\}$, the average over pairwise differences in group means is:

$$\text{divergence} = \frac{2}{G^2 - G} \sum_{i<j}^{G} |\bar{g_i} - \bar{g_j}| \ . \tag{7}$$

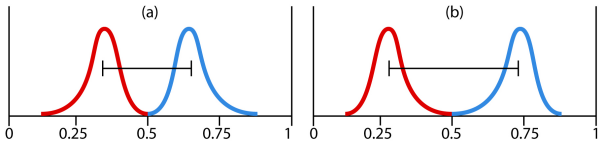

Fig. 8. Attitude distribution (b) shows greater polarization and diversity than (a) in the sense of group divergence.

3.8 Group Solidarity

Often polarization and diversity refer to a whole population and/or differences among the groups within the population, however this sense of polarization and diversity reflects the make-up of the groups themselves. Groups that are more unified in their positions are themselves less diverse and less polarized for the same reasons described in section 3.2 on dispersion. However, when there are multiple groups, and each groups is unified in its belief, then it becomes more difficult to bring them together. More diffuse groups are easier to bring together because even within the groups there are already a variety of ideas. High group solidarity therefore implies greater polarization. Diversity, on the other hand, decreases as the groups become less diffuse. The intuition behind that interpretation of diversity is that having to specialists is less diverse than having two generalists.

Group solidarity for either polarization or diversity can be measured via any of the previously considered measures of population dispersion: aggregated in-group absolute deviations, variances, pairwise distances, or even aggregated group spreads. For a set of G belief sub-distributions (groups) $\{g_1 \dots g_G\}$ with with population $\{n_1 \dots n_G\}$, our example aggregated group solidarity measure is:

$$\text{group solidarity} = \frac{1}{G} \sum_{i=1}^{G} \frac{1}{n_i} \sum_{j=1}^{n_i} |\bar{x_j} - \bar{X_i}| \ . \tag{8}$$

3.9 Size Disparity

So far none of our senses are directly affected by the sizes of the groups in a population. Our formal measures, for example, either ignore groups completely

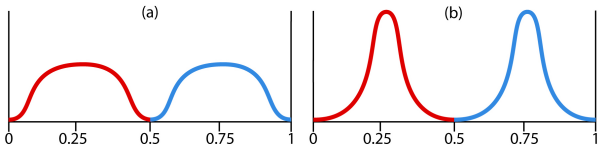

Fig. 9. Distribution (b) shows greater polarization than (a) in the sense of group solidarity, however (a) shows greater diversity than (b).

or focus on in-group aggregates and comparisons of groups means. However it is clear that sub-group sizes do play a role in our umbrella concepts of polarization and diversity. Holding the number of groups constant, a society that has one dominant opinion group with a few small minority outliers is less polarized than one with all similarly sized groups. Although this sense is independent of the particular belief distributions of the groups, the reasoning here is that it is easier to get smaller groups to converge with a large one than it is to get equally sized groups to merge. Diversity also decreases as the size disparity increases; the more equally sized the groups are, the more variation exists in the system because smaller groups represent more marginalized fringe groups.

To measure size disparity independently of the other sense requires a formalism that is sensitive to the sizes of the groups – reaching maximum value when they are all the same – while being simultaneously insensitive to the number of groups. A normalized entropy measure fits this description exactly. For a set of G groups each with population *proportion* $\{p_1 \dots p_G\}$,

$$\text{size disparity}_{\text{entropy}} = -\frac{1}{\ln G} \sum_{i=1}^{G} p_i \ln p_i \ . \tag{9}$$

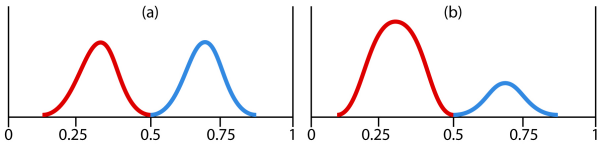

Fig. 10. Groups with comparable sizes are more polarized and diverse than a large group with smaller outlier groups; therefore in the sense of size disparity, attitude distribution (a) shows a more polarized and more diverse population than (b).

3.10 Attitude Association

The above senses of polarization and diversity can be captured via beliefs on one topic (i.e., a distribution along one dimension), although they can be generalized

to beliefs across multiple dimensions. Attitude association requires at least two distinct topics, and it captures how well people's positions on those topics match up. In accordance with the general principle of polarization put forth earlier, it would take more effort to change people's attitude on a topic if it were strongly linked to another topic. This is because it would also require changing their attitude on the associated topic. Therefore, the more closely associated the values on different topics are, the more polarized the overall positions are. However, the general principle of diversity produces the opposite conclusion: highly associated ideas across topics implies fewer variations of idea pairs and hence less diversity.

We measure attitude association using the Pearson correlation coefficient based on the values in the data (the sample). For a population of N individuals with opinions $\{x_1, \ldots, x_N\}$ and $\{y_1, \ldots, y_N\}$ on topics x and y respectively, we can calculate

$$\text{attitude association} = \frac{\sum_{i=1}^{N}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{N}(x_i - \bar{x})^2 \sum_{i=1}^{N}(y_i - \bar{y})^2}} \ . \tag{10}$$

4 Measure Combinations

Capturing the senses independently is vital for understanding the core conceptual basis of how systems might be polarized or diverse. Noting the connections among the measures used for these independent senses is necessary to pierce through the confusion that can occur when applying these senses to data. Furthermore, we can get a deeper understanding of this family of concepts by developing measures that capture multiple senses at once while still keeping the contributions separated. We do not claim that any one of the above senses fully captures any particular notion of polarization, but rather that these are the minimal building blocks of all such notions. And these building blocks can be combined in various ways to capture differing, richer notions of polarization.

Many more combined measures of this type can be devised. Future work will more fully developed the catalog of combined measures. The goal will be to dissect them into their independent senses and determine how the combined measure tracks changes along each sense in isolation and in concert.

5 Conclusions

A clear understanding of any social phenomena demands a clear set of tools for description and analysis. Here we focused on the polarization and diversity of ideas among individuals along a spectrum. That focus reflects our application interests, not a limit to the applicability our breakdown or formalism. The various senses and their measures hold across disciplines for any property distribution. Not all the senses presented are equally strong matches for intuitive notions of polarization, but they each capture a key aspect. A fuller and more nuanced conceptualization of polarization and diversity will combine these independent senses into one, or possibly a few, core general notions.

For most of the senses polarization and diversity increase in parallel. Of the measures covered here, only group solidarity and attitude association produce levels of polarization and diversity that move in opposite directions. This is potentially important because except in those senses, action to decrease polarization will simultaneously decrease diversity. That brings the realization that there are some issues, for example people's rights, in which diversity is undesirable. Diversity implies a variety of beliefs and opinions on topics where universal agreement is actually the goal.

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