Collective decision-making on complex landscapes

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Abstract. A continuous-time Markov process is proposed to analyze how a group of humans solves a complex task, which consists in the search of the optimal set of decisions on a fitness landscape. Individuals change their opinions driven by the self-directed behavior, which pushes them to increase their own fitness value, and by the social interactions, which push individuals to find a common opinion. Results show that increasing the strength of social interactions makes the decision-making process more effective. However, too high values of social interaction strength worsen the performance of the group. We also show that a moderate level of knowledge is already enough to guarantee high performance of the decision-making process.

Keywords: Decision making, social interactions, complexity, Markov chains.

1 Introduction

The ability of groups to solve complex problems that exceed individual skills is widely recognized in natural, human, and artificial contexts. Animals in groups, e.g. flocks of birds, ant colonies, and schools of fish, exhibit collective intelligence when performing different tasks as which direction to travel in, foraging, and defense from predators [1], [2]. Artificial systems such as groups of robots behaving in a self organized manner show superior performance in solving their tasks, when they adopt algorithms inspired by the animal behaviors in groups [3], [4], [5]. Human groups such as organizational teams outperform the single individuals in a variety of tasks, including problem solving, innovative projects, and production issues [6], [7], [8].

The superior ability of groups in solving tasks originates from collective decision making: agents make choices, pursuing their individual goals on the basis of their own knowledge and amount of information, and adapting their behavior to the actions of the other agents. Even though the single agents posses a limited knowledge, and the actions they perform usually are very simple, the collective behavior, enabled by the social interactions, leads to the emergence of a superior intelligence of the group [9], [10], [11].

In this paper we focus on human groups solving complex combinatorial problems. Many managerial problems may be conceived indeed as problems where the effective combinations of multiple and interdependent decision variables should be identified [12], [13], [14], [15]. Our model of collective decision making attempts to capture the main drivers of the individual behaviors in groups, i.e., self-interest and consensus seeking. Agent's choices are made by optimizing the perceived fitness value, which is an estimation of the real fitness value based on the level of agent's knowledge [16], [1]. However, any decision made by an individual is influenced by the relationships he/she has with the other group members. This social influence pushes the individual to modify the choice he/she made, for the natural tendency of humans to seek consensus and avoid conflict with people they interact with [17].

We use the Ising-Glauber dynamics [18] to model the social interactions among group members. The NK model [19], [20] is employed to build the complex fitness landscape associated with the problem to solve. A continuous-time Markov chain is proposed to describe the time evolution of the decision-making process. We define the transition rate of individual's opinion change as the product of the Ising-Glauber rate ([18]), which implements the consensus seeking [22], [21], and an exponential rate [23], which models the self-directed behavior of the individual.

Herein, we explore how both the strength of social interactions and the level of knowledge of the members influence the group performance. We extend previous studies highlighting the efficacy of collecting decision making in presence of a noisy environment [24], and in conditions of cognitive limitations[2], [8].

2 The Model

We consider a human group made of M socially interacting members, which is assigned to solve a complex task. The task consists in solving a combinatorial decision making problem by identifying the set of decisions (choice configuration) with the highest fitness. The fitness function is built employing the NK model [19], [20]. A N-dimensional vector space of decisions is considered, where each choice configuration is represented by a vector $\mathbf{d} = (d_1, d_2, ..., d_N)$. Each decision is a binary variable that may take only two values $+1$ or -1, i.e. $d_i = \pm 1$, $i = 1, 2, ..., N$. The total number of decision vectors is therefore 2^N. Each vector $\mathbf{d}$ is associated with a certain fitness value $V(\mathbf{d})$ computed as the weighted sum of N stochastic contributions $0 \leq W_j \left(d_j, d_1^j, d_2^j, .., d_K^j\right) \leq 1$, each decision leads to a total fitness depending on the value of the decision d_j itself and the values of other K decisions d_i^j, $i = 1, 2, ..., K$. The fitness function of the group is defined as

$$V(\mathbf{d}) = \frac{1}{N} \sum_{j=1}^{N} W_j \left(d_j, d_1^j, d_2^j, .., d_K^j\right) \tag{1}$$

The integer index $K = 0, 1, 2, ..., N-1$ is the number of interacting decision variables, and tunes the complexity of the problem. The complexity of the problem increases with K. Note that, for $K \geq 2$, in computational complexity theory, finding the optimum of the fitness function $V(\mathbf{d})$ is classified as a NP-complete decision problem [25]. This makes this approach particularly suited in our case.

We model the level of knowledge of the k-th member of the group (with $k = 1, 2, ...M$) by defining the $M \times N$ competence matrix $\mathbf{D}$, whose elements D_{kj} take the value $D_{kj} = 1$ if the member k knows that the decision j contributes to the total fitness V, otherwise $D_{kj} = 0$. Based on the level of knowledge each member k computes his/her own perceived fitness (self-interest) as

$$V_k(\mathbf{d}) = \frac{\sum_{j=1}^{N} D_{kj} W_j \left(d_j, d_1^j, d_2^j, .., d_K^j\right)}{\sum_{j=1}^{N} D_{kj}}. \tag{2}$$

Note that if $\sum_{j=1}^{N} D_{kj} = 0$ the perceived fitness is set to zero. Each member of the group makes his/her choices driven by the self-directed behavior, which pushes him/her to increase the self-interest, and by social interactions, which push the member to seek consensus within the group. When $D_{kj} = 0$, for $j = 1, 2, ...N$ the k-th member possesses no knowledge about the fitness function, and his choices are driven only by consensus seeking. Note that the choice configuration that optimizes the perceived fitness Eq. (2), does not necessarily optimize the group fitness Eq. (1). This makes the mechanism of social interactions, by means of which knowledge is transferred, crucial for achieving high-performing decision-making process. We build the matrix $\mathbf{D}$, by randomly choosing $D_{kj} = 1$ with probability $p \in [0, 1]$, and $D_{kj} = 0$ with probability $1-p$. By increasing p from 0 to 1 we control the level of knowledge of the members, which affects the ability of the group in maximizing the fitness function Eq. (1).

All members of the group make choices on each of the N decision variables d_j. Therefore, the state of the k-th member ($k = 1, 2, .., M$) is identified by the N-dimensional vector $\sigma_k = \left(\sigma_k^1, \sigma_k^2, ...\sigma_k^N\right)$, where $\sigma_k^j = \pm 1$ is a binary variable representing the opinion of the k-th member on the j-th decision. For any given decision variable d_j, individuals k and h agree if $\sigma_k^j = \sigma_h^j$, otherwise they disagree. Within the framework of Ising's approach [22], disagreement is characterized by a certain level of conflict E_{kh}^j (energy level) between the two socially interacting members k and h, i.e. $E_{kh}^j = -J\sigma_k^j\sigma_h^j$, where J is the strength of the social interaction. Therefore, the total level of conflict on the entire set of decisions is given by:

$$E = -\sum_j \sum_{(k,h)} J\sigma_k^j\sigma_h^j \tag{3}$$

where the sum on the indexes k and h is over pairs of adjacent spins (every pair is counted once) and the symbol $(\cdot)$ indicates that k and h are nearest neighbors.

A multiplex network [26] with N different layers is defined. On each layer, individuals share their opinions on a certain decision variable d_j leading to a certain level of conflict. The graph of social network on the layer d_j is described in terms of the symmetric adjacency matrix $\mathbf{A}^j$ with elements A_{kh}^j. The interconnections between different layers represent the interactions among the opinions of the same individual k on the decision variables.

In order to model the dynamics of decision-making in terms of a continuous-time Markov process we define the state vector $\mathbf{s}$ of the entire group as $\mathbf{s} = (s_1, s_2, ..., s_n) = \left(\sigma_1^1, \sigma_1^2, ...\sigma_1^N, \sigma_2^1, \sigma_2^2, ...\sigma_2^N, ..., \sigma_M^1, \sigma_M^2, ...\sigma_M^N\right)$ of size $n = M \times N$, and the block diagonal adjacency matrix $\mathbf{A} = \mathrm{diag}\left(\mathbf{A}^1, \mathbf{A}^2, ..., \mathbf{A}^N\right)$. Now let be $P(\mathbf{s}, t)$ the probability that, at time t, the state vector takes the value $\mathbf{s}$ out of 2^n possible states. The time evolution of the probability $P(\mathbf{s}, t)$ obeys the master equation

$$\frac{dP(\mathbf{s}, t)}{dt} = -\sum_l w(\mathbf{s}_l \to \mathbf{s}'_l) P(\mathbf{s}_l, t) + \sum_l w(\mathbf{s}'_l \to \mathbf{s}_l) P(\mathbf{s}'_l, t) \tag{4}$$

where $\mathbf{s}_l = (s_1, s_2, ..., s_l, ..., s_n)$, $\mathbf{s}'_l = (s_1, s_2, ..., -s_l, ..., s_n)$. The transition rate $w(\mathbf{s}_l \to \mathbf{s}'_l)$ is the probability per unit time that the opinion s_l flips to $-s_l$ while the others remain temporarily fixed. Recalling that flipping of opinions is governed by social interactions and self-directed behavior a possible ansatz for the transition rates is

$$w(\mathbf{s}_l \to \mathbf{s}'_l) = \frac{1}{2}\left[1 - s_l \tanh\left(\beta J \sum_h A_{lh} s_h\right)\right] \exp\left\{\beta' \left[\Delta V(\mathbf{s}'_l, \mathbf{s}_l)\right]\right\} \tag{5}$$

In Eq. (5) the pay-off function $\Delta V(\mathbf{s}'_l, \mathbf{s}_l) = \bar{V}(\mathbf{s}'_l) - \bar{V}(\mathbf{s}_l)$, where $\bar{V}(\mathbf{s}_l) = V_k(\sigma_k)$, is simply the change of the fitness perceived by the agent k, when its opinion $s_l = \sigma_k^j$ on the decision j changes from $s_l = \sigma_k^j$ to $s'_l = -\sigma_k^j$. The transition rates $w(\mathbf{s}_l \to \mathbf{s}'_l)$ have been chosen to be the product of the transition rate of the Ising-Glauber dynamics [18], and the Weidlich exponential rate $\exp\left\{\beta' \left[\Delta V(\mathbf{s}'_k, \mathbf{s}_k)\right]\right\}$[23]. Note that Eq. (5) satisfies the detailed balance condition. In Eq. (5) the quantity β is the inverse of the so-called social temperature and is a measure of the chaotic circumstances which lead to a random opinion change. The term β' is related to the degree of uncertainty associated with the information about the perceived fitness (the higher β' the less the uncertainty).

To solve the Markov process Eqs. (4, 5), we employ a simplified version of the exact stochastic simulation algorithm proposed by Gillespie [27], [28]. The algorithm allows to generate a statistically correct trajectory of the stochastic process Eqs. (4, 5).

3 Measuring the performance of the collective decision-making process

The group fitness value Eq. (1) and the level of agreement between the members (i.e. social consensus) are used to measure the performance of the collective-decision making process. To calculate the group fitness value, the vector $\mathbf{d} = (d_1, d_2, ..., d_N)$ needs to be determined. To this end, consider the set of opinions $\left(\sigma_1^j, \sigma_2^j, ..., \sigma_M^j\right)$ that the members of the group have about the decision j, at time t. The decision d_j is obtained by employing the majority rule, i.e. we set

$$d_j = \mathrm{sgn}\left(M^{-1} \sum_k \sigma_k^j\right), \qquad j = 1, 2, ..., N \tag{6}$$

If M is even and in the case of a parity condition, d_j is, instead, uniformly chosen at random between the two possible values ± 1. The group fitness is then calculated as $V[\mathbf{d}(t)]$ and the ensemble average (i.e. the mean over multiple simulation runs) $\langle V(t) \rangle$ is then evaluated. The efficacy of the group in optimizing $\langle V(t) \rangle$ is then calculated in terms of normalized average fitness $\langle V(t) \rangle / V_{\max}$ where $V_{\max} = \max[V(\mathbf{d})]$.

The consensus of the members on the decision variable j is measured as

$$\langle C(t) \rangle = \frac{1}{N} \sum_j \left\langle C^j(t) \right\rangle = \frac{1}{N} \sum_{j=1}^{N} \frac{1}{M^2} \sum_{kh=1}^{M} \left\langle \sigma_k^j(t) \, \sigma_h^j(t) \right\rangle \tag{7}$$

Note that $\left\langle \sigma_k^j(t) \, \sigma_h^j(t) \right\rangle = R_{hk}^j(t)$ is the correlation function of the opinions of the members k and h on the same decision variable j, and $0 \leq \langle C(t) \rangle \leq 1$.

4 Simulation and results

We consider a group of $M = 6$ members which have to make $N = 12$ decisions. For the sake of simplicity, the network of social interactions on each decision layer j is described by a complete graph, where each member is connected to all the others. We also set $\beta' = 10$, since we assume that the information about the perceived fitness function is characterized by a low level of uncertainty. We simulate many diverse scenarios to investigate the influence of the parameter p, i.e. of the level of knowledge of the members, and the effect of the parameter βJ on the final outcome of the decision-making process. For any given p and βJ, each stochastic process Eqs. (4, 5) is simulated by generating 100 different realizations (trajectories). For each single realization, the competence matrix $\mathbf{D}$ is set, and the initial state of the system is obtained by drawing from a uniform probability distribution, afterwards the time evolution of the state vector is calculated with the stochastic simulation algorithm. Fig. 1 shows the time-evolution of normalized average fitness $\langle V(t) \rangle / V_{\max}$ and consensus $\langle C(t) \rangle$, for $p = 0.5$ (i.e. for a moderate level of knowledge of the members), different values of the complexity parameter $K = 1, 5, 11$, and different values of $\beta J = 0.0, 0.5, 1.0$. We observe that for $\beta J = 0$, i.e. in absence of social interactions [see Fig. 1(a)] the decision-making process is strongly inefficient, as witnessed by the very low value of the average fitness of the group. Each individual of the group makes his/her choices in order to optimize the perceived fitness, but, because of the absence of social interactions, he/she behaves independently from the others and does not receive any feedback about the actions of the other group members. Hence, individuals remain close to their local optima, group fitness cannot be optimized and the consensus is low [see Fig. 1(b)] As the strength of social interactions increases, i.e., $\beta J = 0.5$ [Fig. 1(c)], members can exchange information about their choices. Social interactions push the individuals to seek consensus with the member who is experiencing higher payoff. In fact, on the average, those members, which find a higher increase of their perceived fitness, change opinion

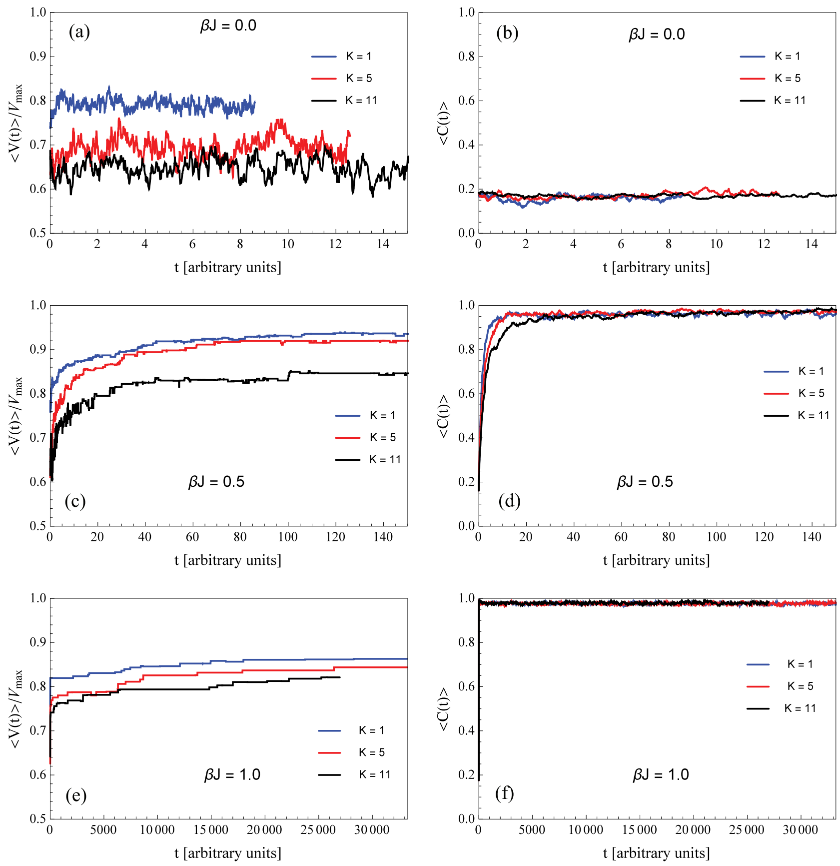

Fig. 1. The time-evolution of the normalized average group fitness, and total consesus, for $p = 0.5$, $K = 1, 5, 11$. $\beta J = 0.0$, (a,b); $\beta J = 0.5$, (c,d); $\beta J = 1.0$, (e,f).

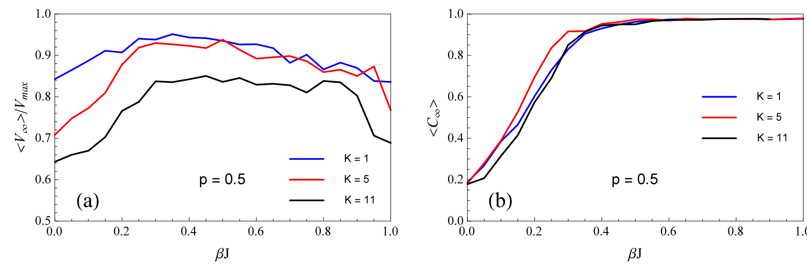

Fig. 2. The stationary values of the normalized averaged fitness $\langle V_\infty \rangle / V_{\max}$ as a function of βJ, (a); and of the statistically averaged consensus $\langle C_\infty \rangle$ as a function of βJ, (b). Results are presented for $p = 0.5$, $K = 1, 5, 11$

much faster than the others. Thus, the other members, in process of seeking consensus, skip the local optima of their perceived fitness and keep exploring the landscape, leading to a substantial increase of the group performance both in terms of group fitness values [Fig. 1(c)] as well as in terms of final consensus [Fig. 1(d)]. Thus, the system collectively shows a higher level of knowledge and higher ability in making good choices than the single members (i.e., a higher degree of intelligence). It is noteworthy that when the strength of social interactions is too large, $\beta J = 1$, [Fig. 1(e)] the performance of the group in terms of fitness value worsens. In fact, very high values of βJ, accelerating the achievement of consensus among the members [Fig. 1(f)], significantly impede the exploration of the fitness landscape and hamper that change of opinions can be guided by payoff improvements.

The search of the optimum on the fitness landscape is slowed down, and the performance of the collective decision-making decreases both in terms of the time required to reach the steady-state as well as in terms of group fitness.

Figure 1, shows that rising the complexity of the landscape, i.e. increasing K, negatively affects the performance of the collective decision-making process, but does not qualitatively change the behavior of the system. However, Figure 1(c) also shows that, in order to cause a significant worsening of the group fitness, K must take very large values, i.e., $K = 11$. Instead, at moderate, but still significant, values of complexity (see results for $K = 5$) the decision-making process is still very effective, leading to final group fitness values comparable to those obtained at the lowest level of complexity, i.e., at $K = 1$.

In Figure 2, the steady-state values of the normalized group fitness $\langle V_\infty \rangle / V_{\max} = \langle V(t \to \infty) \rangle / V_{\max}$ [Fig. 2(a)], and social consensus $\langle C_\infty \rangle = \langle C(t \to \infty) \rangle$ [Fig. 2(b)] are shown as a function of βJ, for $p = 0.5$ and the three considered values of $K = 1, 5, 11$. In particular increasing βJ from zero, makes both $\langle V_\infty \rangle / V_{\max}$ and $\langle C_\infty \rangle$ rapidly increase. This increment is, then, followed by a region of a slow change of $\langle V_\infty \rangle / V_{\max}$ and $\langle C_\infty \rangle$. It is worth noticing, that the highest group fitness value is obtained at the boundary between the increasing and almost stationary regions of $\langle C_\infty \rangle$. Moreover, results show that high consensus is necessary to guarantee high efficacy of the decision-making process, i.e. high values of $\langle V_\infty \rangle / V_{\max}$. Figure. 2(a) also stresses that the fitness landscape complexity (i.e., the parameter K) marginally affects the performance of the decision-making process in terms of group fitness, provided that K does not take too high values. In fact curves calculated for $K = 1, 5$ run close to each-other.

In Figure 3 the steady-state values of the normalized group fitness $\langle V_\infty \rangle / V_{\max}$ [Fig. 3(a)], and social consensus $\langle C_\infty \rangle$ [Fig. 3(b)] are shown as a function of p, for $\beta J = 0.5$ and the three considered values of $K = 1, 5, 11$. Note that as p is increased from zero, the steady state value $\langle V_\infty \rangle / V_{\max}$ initially grows fast [Fig. 3(a)]. In fact, because of social interactions, increasing the knowledge of each member also increases the knowledge of the group as a whole. But, above a certain threshold of p the increase of $\langle V_\infty \rangle / V_{\max}$ is much less significant. This indicates that the knowledge of the group is subjected to a saturation effect. Therefore, a moderate level of knowledge is already enough to guarantee very

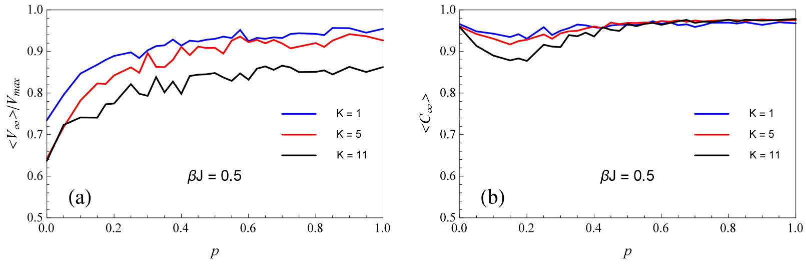

Fig. 3. The stationary values of the normalized average group fitness $\langle V_{\infty}\rangle / V_{\max}$ as a function of p, (a); and of the statistically averaged consensus $\langle C_{\infty}\rangle$ as a function of p, (b). Results are presented for $\beta J = 0.5,\ K = 1, 5, 11$

good performance of decision-making process, higher knowledge levels being only needed to accelerate the convergence of the decision-making process. Figure 3(b) shows that for vanishing values of p the consensus $\langle C_{\infty}\rangle$ takes high values, as each member's choice is driven only by consensus seeking. Increasing p initially causes a decrease of consensus, as the self-interest of each member leads to a certain level of disagreement. However, a further increment of p makes the members' knowledge overlap so that the self-interest of each member almost points in the same direction, resulting in a consensus increase.

5 Conclusions

We have presented a model of collective decision-making on complex landscapes. The model describes the time evolution of group choices in terms of a time-continuous Markov process, where the transition rates have been defined so as to capture the effect of the two main forces, which drive the change of opinion of the members of the group. These forces are the self-directed behavior which pushes each member to increase his/her self-interest, and the social interactions, which push the members to reach a common opinion. Our study identifies under which circumstances collective decision making performs better. We found that a moderate strength of social interactions allows for knowledge transfer among the members, leading to higher knowledge level of the group as a whole. This mechanism, coupled with the ability to explore the fitness landscape, makes the entire group behave as unique entity characterized by a higher degree of intelligence. We also found that increasing the level of knowledge of the members improves performance. However, above a certain threshold the knowledge of the group saturates. Our results also shows that human groups with optimal levels of members' knowledge and strength of social interactions very well manage complex problems.

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